Tentamen Metrische Ruimten
2 februari 2007, 09:00 - 12:00 uur, Examenhal

You can answer the exam in Dutch or English.

1. Consider the subset $H = (0,1)$ of $\mathbb{R}$ with the induced subspace metric. Find the set of limit points and the closure of the following subsets of H. Which of these sets are closed, which are complete and which are compact?
   (a) $(0,1)$.
   (b) $(0,1/2]$.
   (c) $\{1/n : n \in \{2,3,4,\dots\}\}$.
   (d) $\mathbb{Q} \cap (0,1)$.
   $\mathbb{Q}$ is the set of rational numbers. The metric in $\mathbb{R}$ is the standard metric $d(x,y) = |x-y|$. Support your answers by complete arguments.

2. Prove that a finite subset of a metric space M has no limit points.

3. Consider a set A with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Recall that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, if $\mathcal{T}_1 \subset \mathcal{T}_2$. This means that if $U \subset A$ is open with respect to the $\mathcal{T}_1$ topology (i.e. $U \in \mathcal{T}_1$), then U is also open with respect to the $\mathcal{T}_2$ topology (i.e. $U \in \mathcal{T}_2$). The opposite is not true, i.e. there are sets $V \subset A$ that are open in the $\mathcal{T}_2$ topology but not open in the $\mathcal{T}_1$ topology.

   If the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set A satisfy $\mathcal{T}_1 \subset \mathcal{T}_2$ then which of the following statements is correct and which is wrong and why?
   (a) If the set A with the topology $\mathcal{T}_1$ is connected then A with the topology $\mathcal{T}_2$ is connected.
   (b) If the set A with the topology $\mathcal{T}_1$ is not connected then A with the topology $\mathcal{T}_2$ is not connected.

4. Consider two maps $f_1 : T_1 \to S_1$ and $f_2 : T_2 \to S_2$ where S_1, S_2, T_1, T_2 are topological spaces. Define the map

$$f_1 \times f_2 : T_1 \times T_2 \to S_1 \times S_2$$

   such that if $(x,y) \in T_1 \times T_2$ then

$$(f_1 \times f_2)(x,y) = (f_1(x), f_2(y))$$

   Prove that $f_1 \times f_2$ is continuous if and only if f_1 and f_2 are continuous.